ECE Written Qualifying Examination, Spring 2019: Probability

Solutions

1. (7 points) Let X and Y be random variables with joint pdf

$$f_{XY} = cx^2y, \;\; 0 \le x \le y \le 1$$

(a) (1 point) Determine the constant c. Answer:

$$1 = \int_0^1 \left[\int_0^y cx^2y\,dx\right] dy = c/15$$

So $c = 15$.

(b) (1 point) Determine the marginal pdf of X. Answer:

$$f_X(x) = \int_x^1 15x^2y\,dy = (15/2)x^2(1-x^2), \;\; 0 \le x \le 1$$

(c) (1 point) Determine the marginal pdf of Y. Answer:

$$f_Y(y) = \int_0^y 15x^2y\,dx = 5y^4, \;\; 0 \le y \le 1$$

(d) (1 point) Are X and Y independent? Explain. Answer: They are not independent since $f_{XY}(x,y) \ne f_X(x)\,f_Y(y)$.

(e) (1 point) Determine the conditional pdf of X given $Y = y$. Answer:

$$f_{X|Y}(x\,|\,y) = \frac{15x^2y}{5y^4} = \frac{3x^2}{y^3}, \;\; 0 \le x \le y, \; 0 \le y \le 1$$

(f) (1 point) Determine the conditional expectation of X given $Y = y$. Answer:

$$E[X\,|\,Y=y] = \int_0^y x\frac{3x^2}{y^3}\,dx = (3/4)y, \;\; 0 \le y \le 1$$

(g) (1 point) Let A be the event that $XY \le 1/2$. Determine $P(A)$. Answer:

$$\begin{aligned} P(A) &= \int_0^1 \left[\int_0^{\min(y,1/(2y))} 15x^2y\,dx\right] dy \\ &= 5\int_0^1 y\,[\min(y,1/(2y))]^3\,dy \\ &= 5\left[\int_0^{1/\sqrt{2}} y^4\,dy + \int_{1/\sqrt{2}}^1 \frac{1}{8y^2}\,dy\right] \\ &= \frac{1}{8}(6\sqrt{2}-5) \end{aligned}$$

2. (6 points) Let N denote the number of deer in a park. It is known that $N > 25$. Ten of the deer were previously captured, tagged, and released. Suppose that later 20 deer are captured.

   (a) (3 points) Find the probability that 5 of these are found to be tagged. Denote this probability by $p(N)$. **Answer:**

$$p(N) = \frac{\binom{10}{5}\binom{N-10}{15}}{\binom{N}{20}}$$

   (b) (1 point) Determine the ratio $p(N)/p(N-1)$. **Answer:** Using the result in (a) to obtain $p(N)$ and $p(N-1)$, and simplifying, we obtain

$$\begin{aligned} \frac{p(N)}{p(N-1)} &= \frac{(N-10)!(N-1)!(N-26)!(N-20)!}{(N-11)!N!(N-25)!(N-21)!} \\ &= \frac{(N-10)(N-20)}{N(N-25)} \end{aligned}$$

   (c) (2 points) Given the information that 5 were found to be tagged, a reasonable way to estimate N is to choose the value of N that maximizes the observation probability $p(N)$. Using your answer to (b), find this value of N. **Answer:** The ratio in (b) can be expressed as

$$\frac{p(N)}{p(N-1)} = 1 + \frac{-5N+200}{N^2-25N}$$

   Since it is given that $N > 25$, the denominator is always positive. The numerator is positive for $N < 40$, negative for $N > 40$, and 0 for $N = 40$. It follows that $p(N)$ takes its maximum value at $N = 39$ and $N = 40$.

3. (7 points) The output Y of a binary communication system is a unit-variance Gaussian random variable with mean 0 when the input X is 0, and mean 1 when the input is 1. Assume that the input is 1 with probability p.

   (a) (4 points) Find $P[X = 1 \,|\, y < Y < y+h]$ and $P[X = 0 \,|\, y < Y < y+h]$ under the assumption that h is infinitesimally small. **Answer:**

$$\begin{aligned} P(y < Y < y+h \,|\, X = 0) &= \int_y^{y+h} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \approx \frac{1}{\sqrt{2\pi}} e^{-y^2/2} h \\ P(y < Y < y+h \,|\, X = 1) &= \int_y^{y+h} \frac{1}{\sqrt{2\pi}} e^{-(t-1)^2/2} dt \approx \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2} h \end{aligned}$$

   Applying Bayes Rule gives

$$\begin{aligned} P(X = 0 \,|\, y < Y < y+h) &= \frac{\frac{1}{\sqrt{2\pi}} e^{-y^2/2} h(1-p)}{\frac{1}{\sqrt{2\pi}} e^{-y^2/2} h(1-p) + \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2} hp} \\ &= \frac{1-p}{1-p+pe^{y-(1/2)}} \end{aligned}$$

$$P(X=1\,|\,y<Y<y+h) = \frac{\frac{1}{\sqrt{2\pi}}e^{-(y-1)^2/2}hp}{\frac{1}{\sqrt{2\pi}}e^{-y^2/2}h(1-p)+\frac{1}{\sqrt{2\pi}}e^{-(y-1)^2/2}hp}$$
$$= \frac{p}{p+(1-p)e^{-y+(1/2)}}$$

(b) (3 points) The receiver uses the following decision rule: If

$$P[X=1\,|\,y<Y<y+h] > P[X=0\,|\,y<Y<y+h]$$

decide that input was 1; otherwise, decide that input was 0. Show that this decision rule leads to the following threshold rule: If $Y>T$, decide input was 1; otherwise, decide input was 0. Express T as a function of p. If the inputs 0 and 1 are equally likely, what is the corresponding value of T? **Answer:** Substituting the answers from (a) in the decision rule and simplifying, it follows that the decision will be 1 if

$$e^{2y-1} > \frac{(1-p)^2}{p^2}$$

which implies that

$$y > \frac{1}{2} + \log\frac{1-p}{p}$$

So $T = \frac{1}{2} + \log\frac{1-p}{p}$. If the inputs are equally likely, $T=\frac{1}{2}$.