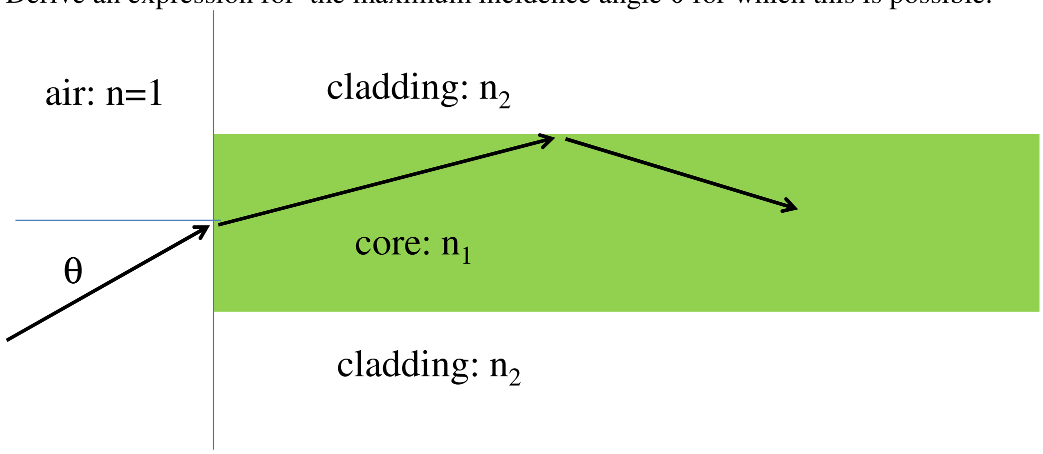
ECE Written Qualifying Examination, Spring 2019 – Electromagnetism

Problems

1. (6pts) Consider an infinitely long wire-wound solenoid oriented along z axis, with radius a, wound with $N \gg 1/a$ turns/meter, and current I in the wire. Find an expressions for the magnetic flux density **B** and the magnetic vector potential **A** inside and outside the solenoid (except for the small region right next to the wire). Some of the vector calculus formulas at the bottom of this page may be useful.

2. (7pts) An optical beam travels from air into a dielectric slab waveguide which has a core with index of refraction n_1 and a cladding with index of refraction n_2. The beam experiences total internal reflection in the waveguide.
Derive an expression for the maximum incidence angle θ for which this is possible.



3. A non-magnetic medium has dielectric constant given by $\varepsilon = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$.

a. (4pts) Derive the expressions for the group velocity and for the phase velocity of a plane wave with $\omega > \omega_p$ which is propagating through this medium.

b. (3pts) Explain what happens when a wave which has frequency $\omega < \omega_p$ is incident on to this medium from vacuum.

Vector calculus operations in cylindrical and spherical coordinates

$$\vec{\nabla} V = \hat{a}_\rho \frac{\partial V}{\partial \rho} + \hat{a}_\phi \frac{\partial V}{\rho \partial \phi} + \hat{a}_z \frac{\partial V}{\partial z} \qquad \vec{\nabla} V = \hat{a}_r \frac{\partial V}{\partial r} + \hat{a}_\theta \frac{\partial V}{r \partial \theta} + \hat{a}_\phi \frac{\partial V}{r \sin\theta \partial \phi}$$

$$\vec{\nabla} \cdot \vec{A} \equiv \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z \qquad \vec{\nabla} \cdot \vec{A} \equiv \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{A_\phi}{\partial \phi}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi \rho & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \qquad \vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{a}_r & \hat{a}_\theta r & \hat{a}_\phi r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

Solutions

1. For a solenoid, $\vec{B}=\mu_0 NI\mathbf{a}_z$ for $\rho \le a$ and $\vec{B}=0$ for $\rho > a$

$$\Psi = \iint_S \vec{B}\cdot d\vec{S} = \iint_S \left(\nabla\times\vec{A}\right)\cdot d\vec{S} = \oint_L \vec{A}\cdot d\vec{l}$$

Since the direction of A is same as that of the current, we only have A_ϕ component.

For $\rho \le a$, $\oint_L \vec{A}\cdot d\vec{l} = 2\pi\rho A_\phi = \Psi = \mu_0 \pi\rho^2 NI \qquad A_\phi = \mu_0 \dfrac{\rho NI}{2}$

For $\rho > a$, $\oint_L \vec{A}\cdot d\vec{l} = 2\pi\rho A_\phi = \Psi = \mu_0 \pi a^2 NI \qquad A_\phi = \mu_0 \dfrac{NIa^2}{2\rho}$

We can also find A by using $\vec{B} = \nabla\times\vec{A}$

Using the expression for the curl in cylindrical coordinates

For $\rho \le a$, $\mu_0 NI\mathbf{a}_z = \dfrac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi\rho & \hat{a}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ 0 & \rho A_\phi & 0 \end{vmatrix} = \dfrac{1}{\rho}\left(\hat{a}_z \dfrac{\partial}{\partial\rho}\rho A_\phi\right) \qquad \dfrac{d}{d\rho}\rho A_\phi = \mu_0 NI\rho \qquad \rho A_\phi = \mu_0 NI\dfrac{\rho^2}{2} + C_1$

We can set this arbitrary constant C_1 to zero. $\underline{A_\phi = \mu_0 NI\dfrac{\rho}{2}}$

For $\rho > a$ $\quad 0 = \dfrac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi\rho & \hat{a}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ 0 & \rho A_\phi & 0 \end{vmatrix} = \dfrac{1}{\rho}\left(\hat{a}_z \dfrac{\partial}{\partial\rho}\rho A_\phi\right) \qquad \dfrac{d}{d\rho}\rho A_\phi = 0 \qquad \rho A_\phi = C_2 \quad A_\phi = \dfrac{C_2}{\rho}$

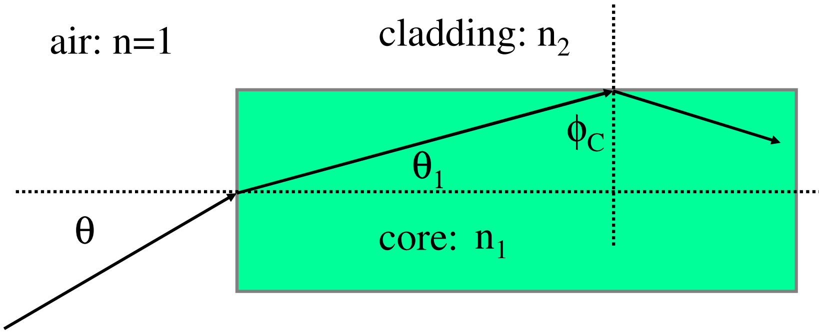
Using continuity of A at ρ=a, we have $A_\phi(a) = \dfrac{C_2}{a} = \mu_0 NI\dfrac{a}{2} \qquad C_2 = \mu_0 NI\dfrac{a^2}{2} \qquad \underline{A_\phi = \mu_0 NI\dfrac{a^2}{2\rho}}$

2. Condition for total internal reflection is: $\sin\phi_C = \dfrac{n_2}{n_1}$

From geometry we have $\phi_C = \dfrac{\pi}{2} - \theta_1$

Snell's law: $\sin\theta = n_1 \sin\theta_1$

$$\sin\theta = n_1 \sin\left(\frac{\pi}{2} - \phi_C\right) = n_1\cos\phi_C = n_1\sqrt{1-\left(\frac{n_2}{n_1}\right)^2} = \sqrt{n_1^2 - n_2^2}$$



3. (7) **a.** $k^2 = \omega^2\mu_0\varepsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right)$ $\qquad v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\varepsilon_0}\sqrt{1-\frac{\omega_p^2}{\omega^2}}} = \frac{c}{\sqrt{1-\frac{\omega_p^2}{\omega^2}}}$

$$v_g = \frac{d\omega}{dk}$$

$$2k\frac{dk}{d\omega} = 2\omega\mu_0\varepsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right) - \omega^2\mu_0\varepsilon_0\left(-2\frac{\omega_p^2}{\omega^3}\right) = 2\omega\mu_0\varepsilon_0\left(\left(1-\frac{\omega_p^2}{\omega^2}\right)+\left(\frac{\omega_p^2}{\omega^2}\right)\right) = 2\omega\mu_0\varepsilon_0$$

$$\frac{dk}{d\omega} = \frac{\omega}{k}\mu_0\varepsilon_0 \qquad v_g = \frac{d\omega}{dk} = \frac{1}{\frac{dk}{d\omega}} = \frac{1}{\frac{\omega}{k}\mu_0\varepsilon_0} = \frac{c^2}{\frac{c}{\sqrt{1-\frac{\omega_p^2}{\omega^2}}}} = c\sqrt{1-\frac{\omega_p^2}{\omega^2}}$$

b. When $\omega < \omega_p$ the dielectric constant is negative and the propagation constant **k** is imaginary. No power is transmitted and an evanescent fields are excited in the medium.